THE CUMULATIVE EXAMINATION IN MATHEMATICS

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Consideration of the comprehensive examination in mathematics has usually encountered the very palpable objection that students are not likely by its means to do themselves justice.

This assertion is to some extent an indictment of present-day methods of teaching, for there can be little question that a comprehensive examination covering three or four years of study is the surest test of the value which a student has received from that study. But few high-school teachers, even granting the impeachment implied in their objection to the comprehensive examination, would, at the present time, be willing to make the passing of such an examination the *sole* criterion of a student's success in mathematics. Too many factors enter into the examination itself. Not the least of these is the general physical and mental condition of the student at the time when the examination is taken. Another is the difficulty of framing a set of questions which will make a reasonable allowance for lapse of memory. A student may readily forget not only theorems and formulae, but in particular cases even proper methods of attack, and still possess a fair knowledge of mathematical principles.

It is, therefore, proposed that the comprehensive (final) examination should not displace the traditional periodical examinations in the various branches of high-school mathematics, but should be *superimposed* upon them. In behalf of the comprehensive examination it is argued that when a student under our present system has passed an examination in algebra, for example, he regards the subject as finished for him. And when, consequently, in geometry or in mechanics, he finds it necessary again to apply algebraic methods, he is pained at the thought of calling up the ghosts of dead ideas. To the serious evils of the "finished subject" any

mathematics teacher, particularly if he has taught the applied branches, can abundantly testify.

But alternatives for the accepted sequence of arithmetic, algebra, plane and solid geometry, and plane trigonometry have met with neither great success nor widespread approval. The reason is not far to seek. The best as well as the poorest of our students organize their knowledge, as they acquire it, largely upon a topical plan. The topical analysis, moreover, of our present mathematical curriculum is not lightly to be cast aside, nor easily to be modified. The arithmetical, the algebraic, the geometrical (Euclidean), and the trigonometrical methods of attacking a problem are usually impressed upon the student's mentality as distinct and separate entities. A mingling of these methods is sure to produce more or less confusion, and the pedagogical problem is to ascertain the extent to which this disadvantage may be offset in point of economy of time, of relative ease of performance, and of immediate applicability, by some arrangement other than the traditional sequence.

To take advantage of this "topical attitude of mind," and at the same time to prepare adequately for a comprehensive final examination, the author submits the plan of a series of cumulative examinations, extending over the three or four years of high-school mathematics. Such examinations, while distinctly characteristic of the subjects just covered in the classroom, should each offer a considerable opportunity for the application of principles brought to light in the preceding term's work.

It is very true that the subjects above referred to are generally recognized to a large extent as mutually interdependent. But it cannot be lost sight of that geometry held its own for many centuries before formal algebra was developed; that many practical surveyors have pursued their calling with very little use for algebra or geometry or knowledge of those subjects; and that many students who excel in the higher branches of mathematics are woefully weak in the art of numbers. These and other considerations of a similar sort have served to maintain the compartment system of examinations, and corresponding compartments in the classroom. Can a single comprehensive final examination break

down the walls, particularly if examinations in the various branches are kept wholly distinct? It is very doubtful. It is more likely, if the comprehensive college-entrance examinations are widely adopted, that its demands will be met by secondary schools with a special course preparatory for the examination itself.

Such a consequence would be lamentable, mathematical teaching being what it is in America today. Altogether too much time is wasted in a rear-end chase between high schools and colleges along the path of examinations. Papers of years past are subjected to the minutest "higher criticism" by high-school teachers for the benefit of their students. College-entrance examiners, believing that their examinations should be distinctly selective, often meet these efforts by greater intricacy in the problems they present on their question papers. It is not so very long ago that an eastern university distinguished itself by requesting aspiring candidates for the Freshman class to define the locus of a point equidistant from two skew lines!

The writer is informed that in foreign countries, where comprehensive examinations are the rule, special preparatory courses are in order, and are commonly to be found. But even in the Continental educational systems the evil of distinguishing between preparation for examinations and preparation for future work in mathematics has long been felt, and as a result, in recent years, emphasis upon the examination has given place to supplementary work in the various branches of mathematics. It should be remembered also that in Continental schools distinction between the various branches is not closely adhered to, as in America.

If the comprehensive college-entrance examination leads only to more examination bickering, it will do more harm than good. But if it brings high-school teachers throughout the country to a realization of the necessary interdependence of the mathematical branches, all will rise and call it blessed. It is to aid in this happy fruition that the cumulative examination is proposed.

However closely the four branches of high-school mathematics are of necessity interrelated, each has its own particular emphasis. And when these branches are *united*, as in many of the courses entitled first-year mathematics, or second-year, or third-year, the

particular emphasis is generally lost, and essential topical distinctions are in this way lost sight of.

Such distinctions, when essential, are in point of *method*, rather than of subject-matter. *Arithmetic* proceeds from the known to the desired in linear fashion. As generally taught, it shows gradually increased knowledge at each step, and employs a narrowly limited number of principles in a particular instance. *Algebra* applies only a few more principles, and these for the most part in a well-defined succession; but in a particular problem it reserves any increase of knowledge until the final dénouement, the obtaining of the value of the unknown letter. The art of (Euclidean) geometry lies essentially in applying many principles simultaneously, or nearly so, to a particular case. This arises from the *two*-dimensional character of the subject and, in the opinion of the writer, makes it a much harder subject for the student than either arithmetic or relatively difficult algebra. It should follow algebra in point of time, for the sake of the advantage to be gained from the greater maturity of the student. Even trigonometry might well precede geometry, the geometric facts involved being taken for granted. The only really new element which trigonometry offers to the student, if method alone is considered, is the use of the tables.

The disregard of these points of emphasis has been responsible for the failure of various "spiral" or "sandwich" methods of dealing with the high-school mathematical problem. There is little likelihood of their prolonged success anywhere, except in certain courses in "shop mathematics" or "applied mathematics," where method is largely lost sight of in the importance of the result obtained.

But once the principle is accepted that the fundamental distinction between the mathematical subjects is a matter of method, the comprehensive examination may serve a very useful purpose in bringing to the front a phase of mathematical study which for many years has been largely neglected in this country. I refer to mathematics as an applied science, or to its essentially applicative character. The word "applied," when associated with mathematics, is generally interpreted as referring to the material sciences.

It does, of course, so refer, but not absolutely. Each branch of mathematics has its applications in another or more of the *mathematical* branches themselves.

If the four branches, arithmetic, algebra, geometry, and trigonometry, are kept distinct in point of time, emphasis upon distinct methods can easily be preserved, and their relative intensity carefully regarded. But if each of these branches is regarded as a field for the application of those preceding, the whole mathematical curriculum acquires a unity which is genuinely organic, and which makes the comprehensive examination practicable and desirable; and the evil of the "finished subject" is thereby minimized. Added gains are that mechanics may be brought into the mathematical curriculum, where it belongs, and that the student applies his mathematics in the laboratory or shop, because it is his habit to look upon mathematics from that point of view.

Applications may, moreover, be distinctly anticipatory in character without any sacrifice of proper emphasis. A striking example of this is afforded in arithmetic, when this subject is included in the high-school curriculum, as it should be. There is no reason whatever why negative and fractional exponents, as well as the simpler cases of radicals, should not be included in such a course in arithmetic. Considerations of incommensurability, couched in precise language, had best be left to the college, or to the graduate school of a university. But careful attention should be given in arithmetic to the practical significance of various degrees of approximation to the values of radicals, and of π.

Moreover, the methods of teaching arithmetic now in great vogue in grammar schools, relying as they do upon resemblances between problems rather than upon the definite statement of fundamental principles, leave to the high school the formulation of those principles. This is best accomplished by the use of letters, sometimes called "algebraic numbers." Arithmetical operations upon literal expressions are easily comprehended, and the *general* character of such expressions may be strongly impressed upon the student. The construction of formulae is the next step, and the evaluation of formulae is easily understood to represent the application of a general arithmetical principle to a particular case.

The parenthesis and the fraction line may be exploited quite extensively in arithmetic if time permits. Arithmetic may borrow from mechanics such topics as the lever, the wheel and axle, the pulley, and specific gravity. The application of various geometric laws is now quite generally taught in arithmetic under the caption "mensuration." This is distinctly anticipatory.

Algebra properly taught should of course place great emphasis upon the equation. But in high-school courses in algebra a larger proportion of time may well be devoted to a distinct exposition of the manner in which algebra aids in the solution of difficult problems in arithmetic. Skill in computation can be maintained by the frequent assignment of problems the date of which are in terms of large numbers or decimals. The *value* of algebra is nowhere more clearly shown than in the *transformation* of formulae.

Algebra applied to geometric problems may anticipate certain well-known laws of geometric figures in a way which adds greatly to the interest of the subject. If algebra is presented in high school from this point of view, even at the sacrifice of artificial intricacy, it gains greatly in life and tone. An examination paper in algebra, then, can easily be made so as to test the student's arithmetical ability as well as his familiarity with algebraic method.

As the *equation* is distinctive of algebra, so is the *formal demonstration* distinctive of geometry. This method of procedure is too difficult for the first year in high school, and may well be kept for the third year, if the curriculum permits. But geometry affords an excellent field for the application of algebraic methods, in spite of the centuries-old tradition of the inviolability of Euclidean form. The status of twentieth-century geometry teaching is rather dubious. Teachers are well, sometimes too well, aware that the axioms and postulates of Euclid lack the fundamental stability which was formerly ascribed to them (though it is doubtful if Euclid himself was similarly minded toward them). It is known now that the beginnings of the logic of plane geometry lie far back of the first proposition of any standard textbook. The obvious inference, then, is that it might be well to assume a much longer list of postulates and so dispense with many hours of drudgery spent in the mastery of propositions which are practically self-

evident to the student. This step once taken affords an opportunity for even greater precision in the definition of terms and in regard for the logical sequence of the parts of the subject, with time to spare.

How shall this time be spent? The answer is immediate: in original exercises. It is the *character* of these exercises which has the greatest interest for mathematical teachers nowadays. It was perfectly natural, of course, that the first of these exercises to be placed before the students should have the character of additional theorems in *Euclid*. Overlooking the fact that Euclid regarded the *sequence* of the theorems as of equal, if not of greater, importance than the manner of the particular demonstration, teachers have inflicted these "riders" mercilessly upon their classes, making theorems and problems more and more intricate, without regard to their proper arrangement, until geometry is frequently reduced to little more than a puzzle book.

Now the complete abandoning of Euclidean method, which seems to many a desirable step, is at least a bold one, considering the twenty-three hundred years during which Euclid has held sway. It may well be asked whether every student should not become familiar with plane geometry as the most marvelous example of human logic ever set forth, so wonderful that Plato regarded it as divine. But two facts should not be overlooked. One is that, in imposing upon demonstrative geometry the modern scientific mania for classification by means of external appearance, we have already damaged the beautiful structure of Euclid very seriously; and the second is that the nicety of argument so characteristic of Euclid always seems to a student a waste of good time, coming as it does *between* the comparatively slipshod methods of arithmetic and algebra, and the immense sweep of college mathematics.

This solution of the difficulty is offered: that the propositions presented should be manipulated with traditional care; but that their number should be reduced to a minimum; that the time thus saved should be devoted to algebraic exercises based on these theorems, as they come, with numerous examples involving considerable computation; and that the final examination in geometry

should demand all three of the distinct methods thus brought into the course.

There is, of course, no logic whatever in the separation of solid and plane geometry. This is done merely because the combined course is too long for one school year. Solid geometry should therefore be treated, under the proposed plan, as plane geometry is treated.

It should be noted, moreover, that many familiar theorems of geometry are in their demonstration closely associated with algebra. These theorems, usually grouped under the heading "numerical properties of triangles," are established much more easily under the form of algebra than under the traditional Euclidean notation. The problem of the determination of the "Golden Section" and that for the approximate computation of π afford admirable illustrations of the application of the principles of quadratic equations. The transformation of a square into a rectangle under given dimensional relations offers a striking comparison of algebraic and geometric methods; and the association of algebraic graphs with geometric loci, as well as that of trigonometric functions with similar right triangles, brings algebra and geometry into direct relation. The algebraic proof of Hero's formula deserves more attention that it usually gets.

The subject of trigonometry is especially significant to the present discussion in that it affords in itself a striking illustration of the way in which the whole high-school course might well be presented. Here arithmetic, algebra, and geometry are all used freely and without prejudice as to their comparative merits. It is the opinion of the writer that trigonometry is fitly the apex of the high-school course; and it is his experience that students who take this subject find it the most satisfying of all of their high-school mathematics. It seems unfortunate that not more than a half-year can usually be devoted to this valuable work. If it were extended on the one hand to include De Moivre's theorem, and on the other to include a wide range of practical applications, there would be little of the preparatory mathematics which it would not exemplify.

It is undeniable that such methods as have been suggested,

though vastly more interesting, would be somewhat more difficult for the student than the present manner, and would require correspondingly greater skill on the part of the teacher. The percentage of students' failures might increase somewhat at first. But the writer contends that this is a part of the cost of the Comprehensive Examination.

Suggested Questions for Cumulative Examination Papers

ALGEBRA THROUGH PROGRESSIONS

1. Show that a number is divisible by three if the sum of its digits is divisible by three.

2. Given

$$k=\frac{l_1-l_0}{l_0(t_1-t_0)}\,.$$

Solve for l_0.

Find l_0 when $l_1=39.27$, $t_1=99.6$, $t_0=0.4$, $k=.000011$.

3. Find by a short method the value of each of the following

a) $(38\tfrac{3}{4})^2-(17\tfrac{1}{4})^2$.

b) $\sqrt{363}-3\sqrt{\tfrac{1}{3}}-5\sqrt[4]{9}$.

c) $\sqrt{h(2r+h)}$, where $r=4000$, $h=\tfrac{1}{10}$.

(Obtain results correct to the second decimal place.)

4. Derive a general formula for obtaining the principal which will amount to a given sum at a given rate in a given time, at simple interest. And find the principal when the time is sixty-four days, the rate 4 per cent, and the amount \$1,540.

5. Find the dimensions, the volume, and the diagonal of a rectangular solid if three faces about one corner have areas of 108 sq. ft., 264 sq. ft., and 198 sq. ft., respectively.

6. If $y=a+\dfrac{b}{c+x}$, find a, b, and c, if $y=41$ when $x=6$, $y=31.91$ when $x=2.1$, and $y=26.45$ when $x=5.4$.

7. A man pays an annual premium of \$164.37 for 20 years on an endowment life insurance policy. What should be his credit with the insurance company, allowing simple interest at 4 per cent on his payments but disregarding insurance cost?

8. At each stroke an air pump withdraws one-tenth of the total quantity of air under a bell jar. If there were 400 cubic inches of air under the bell jar originally, what portion of the original contents remains after eight strokes?

PLANE GEOMETRY

1. Find the number of degrees, minutes, and seconds in each angle of a triangle if the angles are in the ratio of 1, 5, and 7.

2. Prove a law for the sum of the interior angles of a polygon of n sides, and find the value in degrees, minutes, and seconds of each angle of a regular polygon of 48 sides.

3. Illustrate geometrically the algebraic product of $a+b$ and $a-b$ where a and b are lines.

4. Derive a general formula for the number of degrees in the arcs intercepted by (1) two tangents to a circle which make with each other an angle of $a°$; (2) the three sides of an inscribed isosceles triangle whose vertex angle is $b°$.

5. Show how to divide a line in extreme and mean ratio, prove your construction, and find correct to three decimal places the value of the greater segment if the length of the whole line is 10 units.

6. Find the area of a trapezoid if the bases are 15 and 20, and the legs are 29 and 30. State the geometric theorems involved in the solution of this problem.

7. Construct three mutually tangent circles, having given their three lines of centers; and find the diameters of three such circles, if their lines of centers are respectively 4, 5, and 6 inches.

8. Prove that the areas of two similar triangles are proportional to the squares of homologous lines, and find the distance from the vertex of a line drawn parallel to the base of a triangle of altitude 20 which divides the triangle in the ratio of 3:5.